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JUNE '76

A COMPUTER METHOD FOR THE OPTIMUM DESIGN OF LINEAR SYSTEMS

A thesis submitted
In partial fulfilment of the Requirements for
the degree of Master of Technology in
Electrical Engineering

by
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to the
DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

August - 1969

CERTIFICATE

Certified that this work on 'A Computer Method for the optimum design of linear system' has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENT

I wish to acknowledge my graditude to Dr. B. Sarkar for his valuable suggestions and constant inspirations.

ABSTRACT

This work can be basically divided in to two parts. In the first part, a method has been developed for the evaluation of the Integral of Square of the Derivative of Error (ISDE) of time invariant linear system excited by deterministic input. The method is relatively simpler in theory and easier to compute.

And in the second part the design of linear time invariant system has been formulated as a non-linear programming problem and solved by computer using well-known gradient method. A weighted sum of different order of ISDE has been considered as a criterion to be optimised.

CONTENT

CHAPTER - I

EVALUATION OF PERFORMANCE INTEGRAL

1. INTRODUCTION:

Considerable amount of work has been done on the evaluation of different types of integrals involving the error of the linear time invariant system. Most of them[1,2] are based on time domain analysis and involve matrix transformations and Lyapunov's functions.

In this chapter a method has been developed for the evaluation of the integral of the following type

$$\int_{0}^{\infty} \left[\frac{d^k}{dt^k} e(t) \right]^2 dt$$

which is the integral of the square of k th derivative of error, represented as ISDE(k).

The method is based on Parseval's Theorem and hence is a frequency domain analysis. The order of the derivative can be any thing even more than that of the system. One advantage of this method is that once the value of ISDE(k) is evaluated the values of ISDE(i) for i = 0, 1, 2, . . . , k-1 can easily be computed.

2. PARSEVAL'S THEOREM:

Let $Y(s)$ be the Laplace transform of $y(t)$ defined by

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt \qquad \text{Re}\,[s] > \alpha$$

And let $y(t)$ and $Y(s)$ satisfy the following conditions

$$\lim_{t \to \infty} y(t) = 0 \qquad \text{and} \qquad \lim_{s \to \infty} Y(s) = 0$$

Then the following time and frequency domain relation of Parseval's theorem will be true

$$\int_0^\infty y^2(t)\, dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Y(s)\, Y(-s)\, ds \qquad \ldots (1.1)$$

where the region of convergence of the integrand in the right hand side of (1.1) centres along the imaginary axis in complex plane and is of width less than 2α. And hence the contour of integration for (1.1) can be taken along the imaginary axis and closing the entire left half plane.

3. EVALUATION OF ISDE:

Let $X(s)$ be the Laplace transform of the error of the linear system given by

$$X(x) = \frac{P(s)}{Q(s)} = \frac{\sum_{i=0}^{m} p_i s^i}{\sum_{i=0}^{n} q_i s^i} \qquad m \leq n - 1 \qquad (1.2)$$

Let Q(s) be the Hurwitz polynomial. The method will be developed for the evaluation of

$$\mathrm{ISDE}(k) = \int_0^\infty \left[\frac{d^k}{dt^k} x(t) \right]^2 dt \qquad \ldots \qquad \ldots (1.3)$$

where ISDE(k) corresponds to the k th derivative of the error.

3.1 Extensions of Parseval's Theorem:

Let $y(t) = \frac{d^k}{dt^k} x(t)$, then

$$Y(s) = L\left[y(t)\right] = L\left[\frac{d^k}{dt^k} x(t)\right]$$

$$= s^k X(s) - \sum_{i=o}^{k-1} s^i \overset{(k-i-1)}{x}(o)$$

where s^i is the i th power of s and

$$\overset{(k-i-1)}{x}(o) = \lim_{t \to o} \left[\frac{d^{k-i-1}}{dt^{k-i-1}} x(t)\right]$$

Substituting the above, (1.1) gives

$$\mathrm{ISDE}(k) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[s^k X(s) - \sum_{i=o}^{k-1} s^i \overset{(k-i-1)}{x}(o) \right] \times$$

$$\left[(-s)^k X(-s) - \sum_{i=o}^{k-1} (-s)^i \overset{(k-i-1)}{x}(o) \right] ds$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (-1)^k s^{2k} X(s) X(-s)\, ds$$

$$- \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{i=o}^{k-1} (-1)^i s^{k+i} x^{(k-i-1)}(o)\, X(s) \right] ds$$

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{i=o}^{k-1} (-1)^k s^{k+i} x^{(k-i-1)}(o)\, X(-s) \right] ds$$

$$+ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{j=o}^{k-1} \sum_{i=o}^{k-1} s^j x^{(k-1-j)}(o) (-1)^k x^{(k-1-i)}(o) \right] ds$$

The third [redacted] integral in the above expression will vanish, because the integrand doesnot have any pole inside the left half plane and hence it is analytic there. The fourth integral will also vanish because of the following property:

$$\frac{1}{2\pi j} \oint s^i\, ds = 1 \text{ if } i = -1 \text{ and}$$

$$= 0 \text{ if } i \neq -1$$

And hence ISDE(k) can be written as

$$\mathrm{ISDE}(k) = \frac{1}{2\pi j} (-1)^k \int_{-j\infty}^{j\infty} s^{2k} X(s) X(-s)\, ds$$

$$- \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{i=o}^{k-1} (-1)^i s^{k+i} x^{(k-i-1)}(o)\, X(s) \right] ds \qquad \ldots (1.4)$$

3.2 Spectral Factorization[3,4]

Let $F(s)$ be a rational function of complex variable s which has poles on both left and right half of the complex plane. Then $F(s)$ can be seperated into two additive parts, one corresponding to the poles in the left half plane and the other to the right half plane. This process of seperation is known as spectral factorization.

Since in Parseval's Theorem the integrand, which is of the form

$$F(s) = X(s)\,X(-s)$$

has poles in both half of the complex plane to evaluate the integral, the corresponding spectral factored term of the integrand will be needed.

With $X(s)$ as given in (1.2), the spectral factored terms can be written as given below:

$$X(s)\,X(-s) = \frac{P(s)}{Q(s)}\,\frac{P(-s)}{Q(-s)} = \frac{U(s)}{Q(s)} + \frac{U(-s)}{Q(-s)} \qquad \ldots (1.5)$$

where $U(s) = \sum_{i=o}^{n-1} u_i s^i$

the order of $U(s)$ is always $(n-1)$ irrespective of the order of $P(s)$.

In the above expression $U(s)/Q(s)$ can be considered to be the sum of all partial fractions of $X(s)\,X(-s)$ having poles on the left half plane.

The expression in (1.5) can also be written as

$$P(s)\,P(-s) = U(s)\,Q(-s) + U(-s)\,Q(s)$$

On equating the like powers of s from both sides of the above equation a matrix equation of the form

$$\underline{R}\ \underline{p} = 2\,\underline{S}\ \underline{u} \qquad \ldots \qquad \ldots (1.6)$$

can easily be obtained.

where $\underline{R}$ and $\underline{S}$ are $n \times n$ matrices and $\underline{p}$, $\underline{u}$ are $n \times 1$ coloumn vector.

The vector $\underline{u}$ will give the coefficients of $U(s)$ in (1.5).

3.3 <u>Examples:</u>

The following two examples will illustrate the technique involved in spectral factorization.

<u>Example -1</u>

Let

$$X(s) = \frac{P(s)}{Q(s)} = \frac{p_o}{q_o+q_1\,s+q_2 s^2} \quad \text{and} \quad \frac{U(s)}{Q(s)} = \frac{u_o+u_1 s}{q_o+q_1 s+q_2\,s^2}$$

Hence

$$X(s)X(-s) = \frac{p_o}{q_o+q_1 s+q_2\,s^2} \quad \frac{p_o}{q_o-q_1 s+q_2 s^2}$$

$$= \frac{u_o+u_1 s}{q_o+q_1\,s+q_2 s^2} + \frac{u_o-u_1\,s}{q_o-q_1 s+q_2\,s^2}$$

Cross-multiplication gives

$$p_o^{\,2} \;=\; (u_o + u_1\, s)\,(q_o - q_1\, s + q_2\, s^2)$$

$$+ \quad (u_o - u_1 s)\,(q_o + q_1 s + q_2\, s^2)$$

Collecting the coefficients of like powers of s following two equations are obtained;

$$s^0: \quad p_o^{\,2} \;=\; 2q_o\, u_o$$

$$s^2: \quad 0 \;=\; 2q_2\, u_o - 2q_1\, u_1$$

Thus the matrix equation given in (1.6) can be written as

$$\begin{bmatrix} p_o & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_o \\ 0 \end{bmatrix} = 2 \begin{bmatrix} q_o & 0 \\ q_2 & -q_1 \end{bmatrix} \begin{bmatrix} u_o \\ u_1 \end{bmatrix}$$

Example 2

Let $\quad X(s) \;=\; \dfrac{P(s)}{Q(s)} \;=\; \dfrac{p_o + p_1 s + p_2\, s^2}{q_o + q_1 s + q_2 s^2 + q_3 s^3}$

In the similar way as mentioned in example 1, expression (1.6) can be derived. In this case

$$\underline{R} = \begin{bmatrix} p_o & 0 & 0 \\ p_2 & -p_1 & p_o \\ 0 & 0 & p_2 \end{bmatrix} \underline{S} = \begin{bmatrix} q_o & 0 & 0 \\ q_2 & -q_1 & q_o \\ 0 & -q_3 & q_2 \end{bmatrix} \underline{p} = \begin{bmatrix} p_o \\ p_1 \\ p_2 \end{bmatrix} \quad \underline{u} = \begin{bmatrix} u_o \\ u_1 \\ u_2 \end{bmatrix}$$

3.4 Evaluation of $\overset{(m)}{x}(o)$

The principal part of the Laurent Series expansion of X(s) is given by

$$X(s) = \frac{P(s)}{Q(s)} = \sum_{n=1}^{\infty} \frac{d_n}{s^n} \quad \text{where} \quad d_n = \frac{1}{2\pi j} \int_c \frac{X(s)}{s^{-n+1}} \, ds \qquad (1.7)$$

This expansion is valid out side the circle centered at the origin, whose radius is given by

$$r_1 > \max_i |z_i|$$

where z_i is the vector in the complex plane representing the ith zero of Q(s). The contour of integration c can be taken any where out side this circle.

Expansion (1.7) can easily be obtained by long hand division of the numerator P(s) by Q(s). Taking the inverse Laplace transform, (1.7) gives

$$x(t) = \sum_{n=0}^{\infty} \frac{d_{n+1} \, t^n}{n!} \quad \ldots \qquad \ldots (1.8)$$

From (1.8) it is easy to see that

$$\overset{(m)}{x}(o) = d_{m+1} = \lim_{t \to o} \left[\frac{d^m}{dt^m} x(t) \right] \qquad \ldots (1.9)$$

3.5 Evaluation of $\frac{1}{2\pi j}\int_c s^m X(s)\,ds$

The above can be very easily obtained by expanding X(s) in terms of the Laurent Series given by (1.7). Integrating the expansion term by term and noting that

$$\begin{aligned}\frac{1}{2\pi j}\int_c s^i\,ds &= 0 \quad \text{if} \quad i \neq -1 \quad \text{and} \\ &= 1 \quad \text{if} \quad i = -1\end{aligned}$$

The following can be obtained

$$\frac{1}{2\pi j}\int_c s^m X(s)\,ds = d_{m+1} \qquad \ldots \qquad \ldots(1.10)$$

3.6 Evaluation of $\frac{1}{2\pi j}\int_c s^m X(s)\,X(-s)\,ds$

The above integral can be represented by

$$\begin{aligned}\frac{1}{2\pi j}\int_c s^m X(s)\,X(-s) &= \frac{1}{2\pi j}\int_c s^m \frac{U(s)}{Q(s)}\,ds + \\ &+ \frac{1}{2\pi j}\int_c s^m \frac{U(-s)}{Q(-s)}\,ds\end{aligned}$$

The second integral will vanish, if the contour of integration is taken in the left half plane, where the integrand is analytic. And hence

$$\frac{1}{2\pi j}\int_c s^m X(s)\,X(-s)\,ds = \frac{1}{2\pi j}\int_c s^m \frac{U(s)}{Q(s)}\,ds$$

Let the Laurent Series expansion around the origin of U(s)/Q(s) be given by

$$\frac{U(s)}{Q(s)} = \sum_{n=1}^{\infty} \frac{C_n}{s^n} \qquad C_n = \frac{1}{2\pi j} \int_c \frac{U(s)}{Q(s)}\, s^{n-1}\, ds \qquad (1.11)$$

And hence by the same way as mentioned in section 3.5

$$\frac{1}{2\pi j} \int_c s^m X(s)\, X(-s)\, ds = C_{m+1} \qquad \ldots (1.12)$$

3.7 Value of ISDE(k)

The expressions so far derived can now be used to evaluate the expression for ISDE(k) given in (1.4). Thus

$$\begin{aligned} \mathrm{ISDE}(o) &= C_1 \\ \mathrm{ISDE}(k) &= (-1)^k C_{2k+1} - \sum_{i=o}^{k-1} (-1)^i\, d_{k-i}\, d_{k+i+1} \quad \text{for } k \geq 1 \end{aligned} \qquad \ldots (1.13)$$

where C's and d's are given by (1.11) and (1.7).

Thus,

$\mathrm{ISDE}(o) = C_1$

$\mathrm{ISDE}(1) = -C_3 - d_1\, d_2$

$\mathrm{ISDE}(2) = C_5 + d_1\, d_4 - d_2\, d_3$

$\mathrm{ISDE}(3) = -C_7 - d_1\, d_6 + d_2\, d_5 - d_3\, d_4$

CHAPTER - II

FORMULATION OF THE PROBLEM

1. INTRODUCTION

The classical method of system design is based on trial and error technique and depends very much on the experience of the designer. The possibility of formulating the design procedure as a non-linear programming problem has been mentioned by Newton[14]. But the appearance of the actual formulation in the literature is relatively recent[5,6].

The formulation presented here is a more general one. In this method the error of the system is expressed in terms of the Laplace transform form, as a ratio of two polynonials, and the weighted sum of the ISDE(k) for different values of k, is minimized by selecting proper values of the co-efficients of the above two polynomials. The order of the denominator polynomial should only be given and the numerator will automatically be fixed up by the method.

2. PERFORMANCE INDEX:

For the optimum design of linear systems many types of performance indices have been developed. Unfortunately none of them are [redacted] suitable and very general for the design. A detailed discussion of the different types of performance indices have been discussed in many literature[7,8].

In this design method the system error is represented in its Laplace transform form as

$$X(s) = E(s) = \frac{\sum\limits_{i=o}^{m} p_i \; s^i}{\sum\limits_{i=o}^{n} q_i s^i} \qquad m \leq n - 1$$

with $e(t)$ as its time function. The following performance index is proposed here

$$J = \sum_{k=o}^{M} w_k \;\; ISDE(k) = \sum_{k=o}^{M} w_k \int_{o}^{\infty} \left[\frac{d^k}{dt^k} \; e(t) \right]^2 dt \qquad \ldots (2.1)$$

The above can be found out by the method presented in Chapter 1.

Suitable weights w_k can be selected to give proper stress to the desired derivative of the error. The value of M depends on the choice of the designer.

3. CONSTRAINTS:

i) The derivation of the performance index, given in Chapter 1, shows that (2.1) is a function of $\bar{C}$'s and $\bar{d}$'s which in turn are functions of $\underline{p}$, $\underline{q}$ and $\underline{u}$, where the coloumn vectors $\underline{p}$, $\underline{q}$ and $\underline{u}$ constitute the co-efficients of the numerator and denominator of the Laplace transform of the error and that of the numerator of the spectral factored term, respectively, and is given by (1.2) and (1.5).

Thus performance index (2.1) can be written as

$$J = f(\underline{p}, \underline{q}, \underline{u}) \quad \ldots \quad \ldots (2.2)$$

where $\underline{p}$ is an n-vector, $\underline{q}$ is an (n+1) vector and $\underline{u}$ an n-vector where as f is a scalar function. (n being the order of the system).

In order to define $\underline{u}$ in (2.2) the expression given in (1.6) will have to be satisfied. To take care of this, (1.6) will be considered as a set of equality constraint on the design method, and will be represented in vector matrix form

$$\underline{h}(\underline{p}, \underline{q}, \underline{u}) = \underline{0} \quad \ldots \quad \ldots (2.3)$$

where the function $\underline{h}$ is an n-vector.

ii) The basic requirement of the system is its stability. The error function of the system is given by

$$E(s) = \frac{\sum_{i=o}^{m} p_i s^i}{\sum_{i=o}^{n} q_i s^i} \qquad m \leq n - 1$$

To see that E(s) be stable and minimum phase type both the denominator and numerator polynomial must satisfy the Routh criteria for stability.

To consider this, the first column of the Routh table will be evaluated, corresponding to the numerator and denominator polynomials and set them to be greater than zero. These constraints can be defined as

$$\begin{array}{lll} \underline{r}_1 (\underline{p}) > \underline{0} & \underline{r}_1 = \text{ n vector} & \\ & & \ldots (2.4) \\ \underline{r}_2 (\underline{q}) > \underline{0} & \underline{r}_2 = (n+1) \text{ vector} & \end{array}$$

iii) In the minimization procedure, to avoid the trivial solution, p_n will be set equal to unity and

$$\sum_{i=o}^{n} p_i \geq 1 \qquad \ldots \qquad \ldots (2.5)$$

The inequality constraints given by (2.4) and (2.5) can all be accomodated and represented in the vector matrix form

$$\underline{g} (\underline{p}, \underline{q}) \geq \underline{0} \qquad \ldots \qquad \ldots (2.6)$$

which may include any other extra inequality constraints required for the design, and accordingly the order of the coloumn vector $\underline{g}$ will change.

4. THE PROBLEM

The design of the linear system can now be mentioned as below:

Find out the suitable values for $\underline{p}$'s and $\underline{q}$'s of the Laplace transform of the error given by

$$E(s) = \frac{\sum_{i=o}^{m} p_i s^i}{\sum_{i=o}^{n} q_i s^i} \qquad m \leq n-1$$

so that the performance index

$$J = f(\underline{p}, \underline{q}, \underline{u})$$

is minimized subject to the constraints

$$\underline{g}(\underline{p}, \underline{q}) \geq \underline{0} \quad \text{and}$$

$$\underline{h}(\underline{p}, \underline{q}, \underline{u}) = \underline{0}$$

where $\underline{p} \leq n$ vector, $\underline{u} = n$ vector, $\underline{q} = (n+1)$ vector, $\underline{h} = n$ vector.

The above is the general form of the Non-linear programming problem and can be solved by any of the well known techniques. Here in this thesis the solution has been carried out by the Gradient Method.

5. THE SOLUTION:

To take care of the inequality constraints, these will be added to the cost function in such a way that if any solution point does not violate a constraint, then that constraint will be set equal to zero and hence its contribution to the cost function will be nil, on the other hand if it violates any constraint then the square of the violated constraint will be added to the cost function with a high penalty attached to it. Thus the new cost function can be represented as

$$J' = f(\underline{p}, \underline{q}, \underline{u}) + \frac{1}{R_k} \sum_i H_i(g_i)\, g_i^2(\underline{p}, \underline{q}) \quad \ldots (2.7)$$

where $H_i(g_i) = 0$ if $g_i(\underline{p}, \underline{q}) \geq 0$ and

$= 1$ if $g_i(\underline{p}, \underline{q}) < 0$

and R_k = a small positive quantity.

For a particular value of R_k the minimization procedure is carried out, and stopped if the final solution does not violate any of the inequality constraints. But

if the solution violates any constraint then the value of R_k is reduced and the computation is repeated. It has been shown[9] that the proper solution will be obtained for a sequence of values of R_k tending to zero.

So far as the equality constraints are considered they are always kept satisfied during computation. To start the computation, some arbitrary values of $\underline{p}$, $\underline{q}$ are assumed and the constraints $\underline{h}\ (\underline{p}, \underline{q}, \underline{u}) = 0$ are solved for $\underline{u}$. Then $\underline{p}$, $\underline{q}$ are modified by the gradient method and modified $\underline{u}$ is obtained by solving the equality constraint, and the process is repeated.

6. SELECTION OF STEP SIZE IN THE GRADIENT METHOD

To find out the proper step size in gradient method is a difficult problem. An improper step size may lead to instability or a large number of iterations and hence long computation time.

A method for selecting the step size is proposed here. This method will lead to the optimum point from the assumed initial point in the least number of iterations.

In gradient method, at any initial point the gradient of the objective function is found out, and moved along that gradient upto a distance equal to the step size, and then gradient is found out at that point and the process is repeated untill the optimum is reached.

Figure 2.1 shows the contour of the function to be minimized, in the 'co-ordinate space'. The gradient of the function at any point is normal to the contour of the function through that point. This gradient is also tangent to some contour of lower value. The value of the contour to which the gradient is tangent is the least value of the function that can be obtained by moving along this gradient. The distance of the point of tangency from the initial point is the optimum distance along this gradient which will give the maximum reduction in the cost function. This length will be the step size for the current iteration.

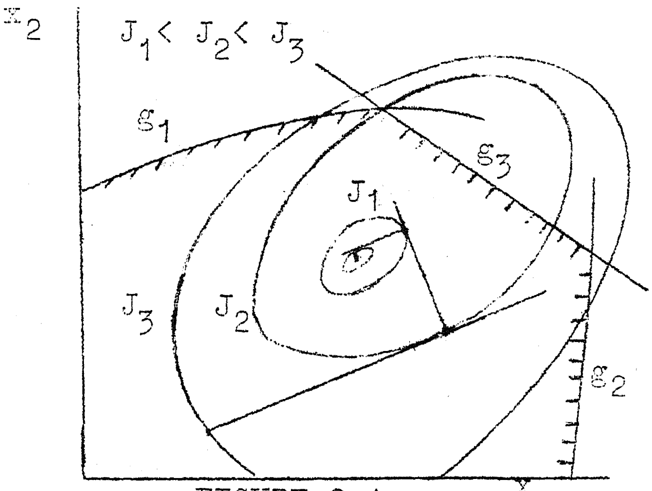

FIGURE 2.1

So the step size at every iteration will be found out by moving along the gradient as long as the value of the cost function decreases and the constraints are not violated. The point at which the cost function will start increasing or the solution will violate any constraints will be the initial point for the next iteration i.e. a new gradient will be found out at this point and the process will be repeated.

CHAPTER - III

EXAMPLES AND RESULTS

1. EVALUATION OF ISDE(k)

$$\text{Let} \quad X(s) = \frac{P(s)}{Q(s)} = \frac{2s + 5}{s^2 + 5s + 6}$$

For this X(s) the equation given in (1.6) becomes

$$\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 6 & 0 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Solving the above, $\underline{u}$ is obtained as

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 2.08 \\ 0.816 \end{bmatrix}$$

and hence spectral factored term corresponding to the poles in the left half plane is given by

$$\frac{U(s)}{Q(s)} = \frac{0.816s + 2.08}{s^2 + 5s + 6}$$

Thus

$C_1 = 0.8167$ $\quad C_4 = -13.50$ $\quad C_7 = 296.1$

$C_2 = -2.00$ $\quad C_5 = 36.90$ $\quad C_3 = 5.1$

$C_6 = -103.50$ $\quad d_1 = 2.00$ $\quad d_2 = -5.00$

$d_3 = 13.00$ $\quad d_4 = -35.00$ $\quad d_5 = 97.00$ $\quad d_6 = -275.00$

and hence

$$ISDE(o) = C_1 = 0.8167$$

$$ISDE(1) = -C_3 - d_1\, d_2 = 4.90$$

$$ISDE(2) = C_5 + d_1\, d_4 - d_2\, d_3 = 31.9$$

$$ISDE(3) = -C_7 - d_1\, d_6 + d_2\, d_5 - d_3\, d_4 = 224.0$$

The above result can easily be veryfied by the following direct computation

$$X(s) = \frac{2s + 5}{s^2 + 5s + 6} = \frac{1}{s + 2} + \frac{1}{s + 3}$$

Therefore $x(t) = e^{-2t} + e^{-3t}$ and hence

$$\dot{x}(t) = -2e^{-2t} - 3e^{-3t}$$

$$\ddot{x}(t) = 4e^{-2t} + 9e^{-3t}$$

$$\dddot{x}(t) = -8e^{-2t} - 27e^{-3t}$$

Now $$\int_0^\infty \left[ae^{-\alpha_1 t} + be^{-\alpha_2 t} \right]^2 dt = \frac{a^2}{2\alpha_1} + \frac{b^2}{2\alpha_2} + \frac{2ab}{\alpha_1 + \alpha_2}$$

Therefore $$ISDE(o) = \int_0^\infty x^2 dt = \frac{1}{4} + \frac{1}{6} + \frac{2}{5} = 0.817$$

$$ISDE(1) = \int_0^\infty \dot{x}^2 dt = \frac{4}{4} + \frac{9}{6} + \frac{12}{5} = 4.9$$

$$ISDE(2) = \int_0^\infty \ddot{x}^2 dt = \frac{16}{4} + \frac{81}{6} + \frac{72}{5} = 31.9$$

$$ISDE(3) = \int_0^\infty \dddot{x}^2\, dt = \frac{64}{4} + \frac{27.27}{6} + \frac{16.27}{5} = 224.0$$

2. DESIGN OF A UNIT NUMERATOR SYSTEM

This article presents the design of a unit numerator third order system. The block diagram configuration is shown in Figure 3.1.

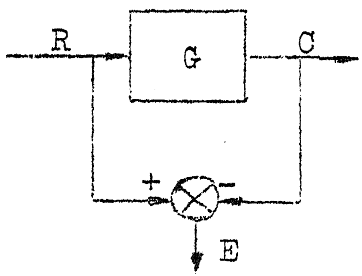

Figure 3.1

Let the system transfor function be given by

$$G(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + 1}$$

suitable values of a_1 and a_2 are needed so that the following is minimized:

$$J = \sum_{k=o}^{M} w_k \int_o^{\infty} \left[\frac{d^k}{dt^k} e(t) \right]^2 dt \qquad \ldots (3.1)$$

where e(t) is the inverse Laplace transform of E(s). For the configuration shown in Figure 3.1.

$$E(s) = R(s) - C(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - G(s)$$

$$= \frac{s^3 + a_1 s^2 + a_2 s}{s^3 + a_1 s^2 + a_2 s + 1}$$

Hence for a step input of unit amplitude

$$E(s) = \frac{s^2 + a_1 s + a_2}{s^3 + a_1 s^2 + a_2 s + 1}$$

Comparing the co-efficients with

$$E(s) = \frac{\sum_{i=o}^{m} p_i s^i}{\sum_{i=o}^{n} q_i s^i} \qquad m \leq n - 1 \qquad \ldots (3.2)$$

following additional constraints are obtained

$q_o = 1 \quad p_o = q_1 \quad p_2 = 1 \quad q_3 = 1 \quad p_1 = q_2$

The results of the optimum system for different values of M and $w_i = 1$ for all i, are presented in Table 3.1.

TABLE 3.1

M	J	a_1	a_2
0	1.5366	1.3450	1.7995
2	2.1576	2.2589	2.1608
4	4.1564	2.0670	2.0921

It is to be noted here that the values of a_1 and a_2 obtained for M = 0 is the same as reported earlier.[7] The transient response of the optimum system for an unit step input is given in Fig. 3.3.

3. DESIGN OF A GENERAL THIRD ORDER SYSTEM

For the similar configuration as given in Fig.3.1 a general third order system can be designed.

$$\text{Let} \quad G(s) = \frac{b_o s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\frac{E(s)}{R(s)} = \frac{s^3 + (a_1 - b_o) s^2 + (a_2 - b_1) s + (a_3 - b_2)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Let $a_3 = b_2$, then for a unit step input

$$E(s) = \frac{s^2 + (a_1 - b_o) s + (a_2 - b_1)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Comparing this with (3.2) the additional constraints obtained are

$$q_3 = 1 \quad \text{and} \quad p_2 = 1$$

The values of a's and b's are given by

$$a_1 = q_2 \qquad a_2 = q_1 \qquad a_3 = q_o$$

$$b_o = q_2 - p_1 \quad b_1 = q_1 - p_o \quad b_2 = q_o$$

The optimum values of the $\bar{a}$'s and $\bar{b}$'s are given in Table 3.2 for different M and with $w_i = 1$ for all i.

The normalized transient response for a step input is given in Fig. 3.4.

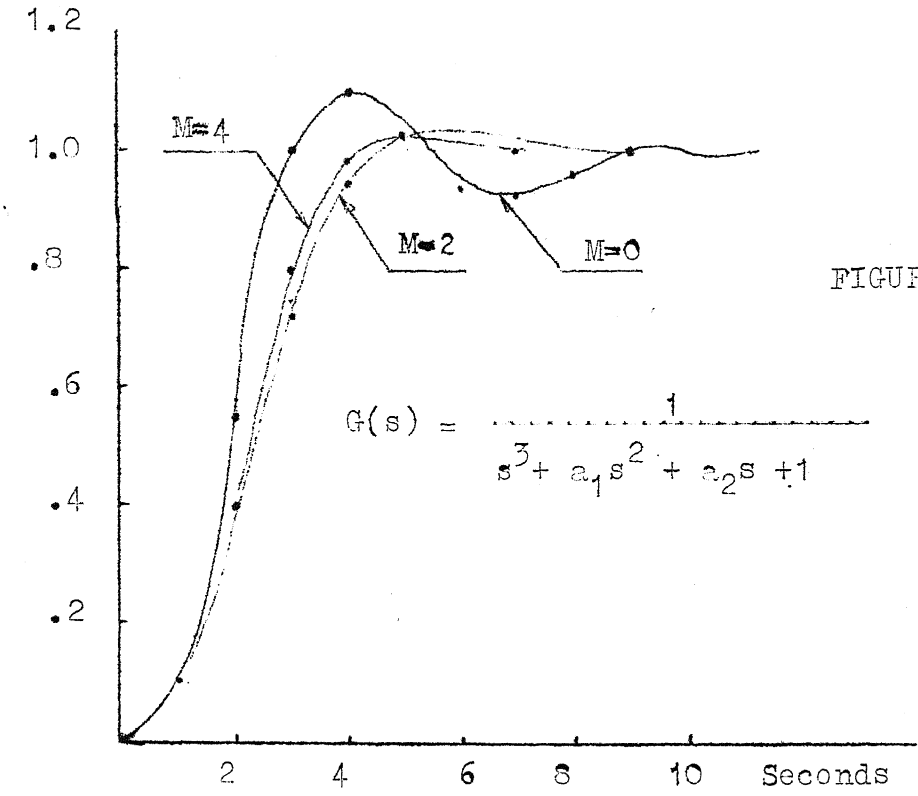

FIGURE 3.3

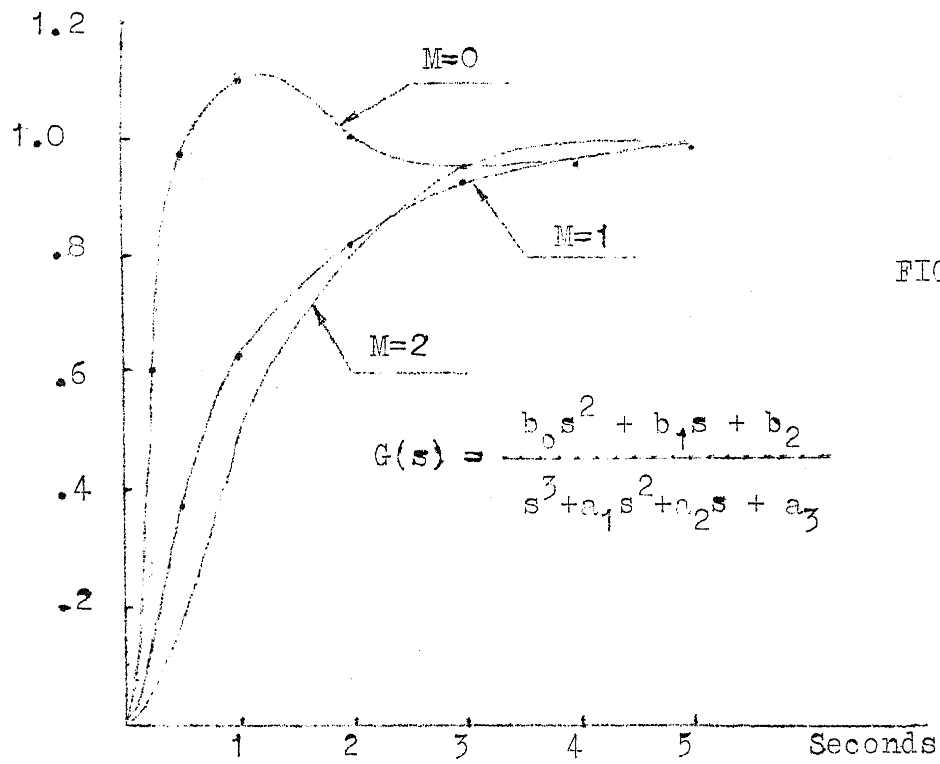

FIGURE 3.4

TABLE - 3.2

M	J	b_o	b_1	b_2	a_1	a_2	a_3
0	0.116	4.30	4.25	2.05	4.30	4.70	2.05
1	1.000	1.00	2.59	1.65	3.60	4.25	1.65
2	1.155	0.58	2.05	1.65	3.50	3.96	1.65

4. THE DESIGN OF A COMPENSATOR

The design technique developed in chapter two can also be extended for the design of a compensator of a control system. The block diagram of the system is given in Fig.3.2 where $G_p(s)$ is the given plant transfer function, and $G_c(s)$ is the compensator transfer function which is required to be designed.

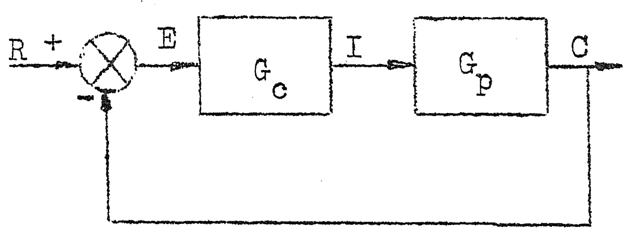

FIGURE 3.2

$$\text{Let} \quad G_p(s) = \frac{\sum_{i=o}^{m} b_i s^i}{\sum_{i=o}^{n} a_i s^i} \qquad m \leq n \text{ and } a_o = 0 \qquad \ldots (3.3)$$

Note that $a_o = 0$ will be needed here for the design, this condition is generally satisfied in most of the physical plants.

Let

$$G_c(s) = \frac{\sum_{i=o}^{r} d_i s^i}{\sum_{i=o}^{k} C_i s^i} \quad r \leq k \qquad \ldots (3.4)$$

Therefore

$$G(s) = G_c(s)\, G_p(s) = \frac{\sum_{i=o}^{m} b_i s^i \;\; \sum_{i=o}^{r} d_i s^i}{\sum_{i=o}^{n} a_i s^i \;\; \sum_{i=o}^{k} C_i s^i}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{\sum_{i=o}^{n} a_i s^i \;\; \sum_{i=o}^{k} C_i s^i}{\sum_{i=o}^{n} a_i s^i \;\; \sum_{i=o}^{k} C_i s^i + \sum_{i=o}^{m} b_i s^i \;\; \sum_{i=o}^{r} d_i s^i} \qquad \ldots (3.5)$$

And hence for a unit step input

$$E(s) = \frac{\sum_{i=o}^{n+k-1} p_i s^i}{\sum_{i=o}^{n+k} q_i s^i} \qquad \ldots (3.6)$$

Thus in this case the additional constraint is that

$$\sum_{i=o}^{n+k} q_i s^i - \sum_{i=o}^{n+k-1} p_i s^i \quad \text{should have} \quad \sum_{i=o}^{m} b_i s^i$$

as a factor, i.e. remainder after dividing by $\Sigma\, b_i s^i$ should be equal to zero or less than some small preassigned tolerance value. Any constraint equivalent to this can also be used.

It is also desirable that the input excitation does not saturate the plant. For this, following constraint is posed on $i(t)$ (shown in Fig. 3.2), the inverse Lapalce transform of $I(s)$:

$$J_o = \int_o^\infty i^2(t)\; dt \leq I_o \qquad \ldots (3.7)$$

The above can be evaluated by the help of Parseval's Theorem knowing the transform of $i(t)$.

Now $I(s) = E(s) \; G_c(s)$

$$= \frac{\sum_{i=o}^{n+k-1} p_i s^i \quad \sum_{i=o}^{r} d_i s^i}{\sum_{i=o}^{n+k} q_i s^i \quad \sum_{i=o}^{k} C_i s^i} \qquad \ldots (3.8)$$

To evaluate the left hand side of (3.7) $G_c(s)$ will have to be found out with the help of (3.5) and (3.6).

The following section gives an example of the compensator design problem.

5. AN EXAMPLE OF COMPENSATOR DESIGN:

Let $G_p(s) = \dfrac{4}{s(s/2+1)} = \dfrac{1}{0.125s^2 + .25s}$

and $G_c(s) = \dfrac{d_1 s + d_o}{C_1 s + C_o}$

The optimum values of d's and C's are needed so that (3.1) is minimized satisfying (3.7).

$$G(s) = G_c(s)\,G_p(s) = \frac{d_1 s + d_o}{.125\,C_1 s^3 + (.125\,C_o + .25C_1)s^2 + .25C_o s}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{.125C_1 s^3 + (.125C_o + .25C_1)s^2 + .25C_o s}{.125C_1 s^3 + (.125C_o + .25C_1)s^2 + (.25C_o + d_1)s + d_o}$$

Thus for unit step

$$E(s) = \frac{.125C_1 s^2 + (.125C_o + .25C_1)\,s + .25C_o}{.125C_1 s^3 + (.125C_o + .25C_1)s^2 + (.25C_o + d_1)\,s + d_o}$$

Thus the additional constraints can be written as

$$p_1 = q_2 \qquad p_2 = q_3 = 1$$

$$q_2 = .125C_o + .25C_1 \quad \text{or} \quad p_o = 2q_2 - 4$$

And hence the parameters of the compensator is given by

$$c_1 = 8 \qquad d_o = q_o$$

$$c_o = 4\,p_o \qquad d_1 = q_1 - p_o$$

The optimum values of c's and d's for M = 1 and 2 and for $w_i = 1$ for all i is given in Table 3.3.

TABLE 3.3

M	J	J_o	d_o	d_1	c_o	c_1
1	1.62	0.14	59.56	27.75	10.76	8
2	5.41	0.12	59.49	-2.02	130.04	8

The normalised transient response for a unit step input is given in Fig. 3.5.

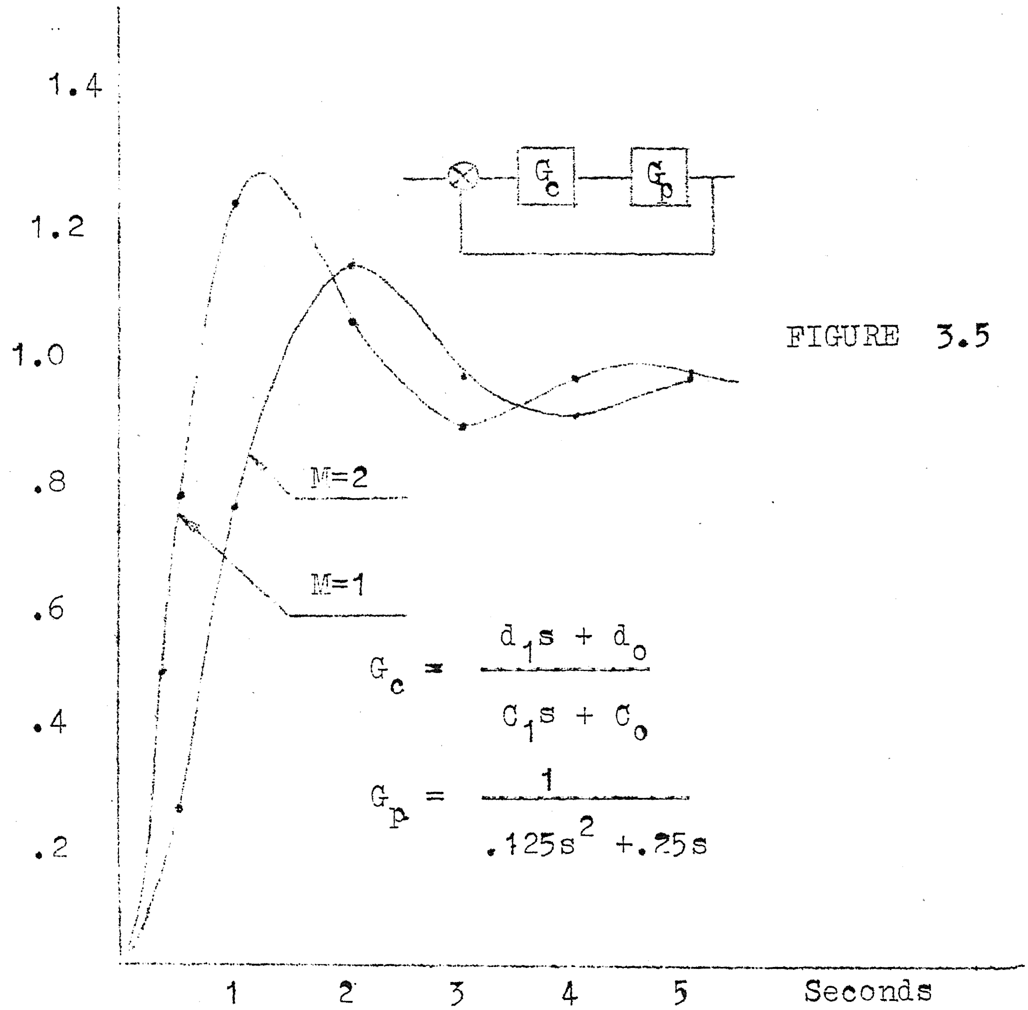
1.4
1.2
1.0
.8
.6
.4
.2
1
2
3
4
5
Seconds
M=2
M=1
G_c
G_p
$G_c = \frac{d_1 s + d_o}{C_1 s + C_o}$
$G_p = \frac{1}{.125s^2 + .25s}$

FIGURE 3.5

CONCLUSIONS

A complete numerical method has been presented for the optimum design of linear time invariant system. The advantage of the method is that it does not need any experience of the designer. Any arbitrary choice of initial parameter of the system can be made to converge to the final optimal value.

A new performance measure has been proposed for the optimum design of linear system. From the transient response given in Fig. 3 3, 3.4 and 3.5 it is observed that the criterion considered is a reasonable one. With proper choice of weights for the ISDE(k) transient response can be given proper shape.

Although the design procedure consists of large number of computations at every iterations, this is not a problem now a days because of the availability of high speed computers.

It has not been possible to insert a suitable, simple criterion which will assure that the optimum system will be controllable.

As a further research in this subject following two important problems are presented.

1) Given two polynomials $\Sigma a_i x^i$ and $\Sigma b_i y^i$ find out a simple relation involving a's and b's so that the above polynomials will not have any common root.

2) A detailed study of the cost function for different values, both negative and positive, of w_i, the weights used in the cost function. Also find out a method to select optimum values of w_i for a particular value of M.

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